MODULES COFINITE AND WEAKLY COFINITE WITH RESPECT TO AN IDEAL

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Abstract. The purpose of the present paper is to continue the study of modules cofinite and weakly cofinite with respect to an ideal $\mathfrak{a}$ of a Noetherian ring R. It is shown that an R-module M is cofinite with respect to $\mathfrak{a}$, if and only if, $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \leq \operatorname{cd}(\mathfrak{a}, M) + 1$, whenever $\dim R/\mathfrak{a} = 1$. In addition, we show that if M is finitely generated and $H^i_{\mathfrak{a}}(M)$ are weakly Laskerian for all $i \leq t-1$, then $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite for all $i \leq t-1$ and for any minimax submodule K of $H^t_{\mathfrak{a}}(M)$, the R-modules $\operatorname{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)/K)$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)/K)$ are finitely generated, where t is a non-negative integer. Finally, we explore a criterion for weakly cofiniteness of modules with respect to an ideal of dimension one. Namely for such ideals it suffices that the two first Ext-modules in the definition for weakly cofiniteness are weakly Laskerian. As an application of this result we deduce that the category of all $\mathfrak{a}$-weakly cofinite modules over R forms a full Abelian subcategory of the category of modules.

1. Introduction

Let R denote a commutative Noetherian ring (with non-zero identity) and $\mathfrak{a}$ an ideal of R. Also, we let M denote an arbitrary R-module.

It is well-known result that if R is a local (Noetherian) ring with maximal ideal $\mathfrak{m}$, then the R-module M is Artinian if and only if $\operatorname{Supp}(M) \subseteq \{\mathfrak{m}\}$ and $\operatorname{Ext}^j_R(R/\mathfrak{m}, M)$ is finitely generated for all $j \geq 0$ (cf. [16, Proposition 1.1]).

Using this idea, Hartshorne [16] introduced the class of cofinite modules, answering in negative a question of Grothendieck (cf. [15, Exposé XIII, Conjecture 1.1]). In fact, Grothendieck conjectured that for any ideal $\mathfrak{a}$ of R and any finitely generated R-module M, the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is finitely generated, where $H^i_{\mathfrak{a}}(M)$ is the i-th local cohomology module of M with support in $V(\mathfrak{a})$, (this is the case when $\mathfrak{a} = \mathfrak{m}$, the maximal ideal in a local ring, since the modules $H^i_{\mathfrak{m}}(M)$ are Artinian), but soon Hartshorne was able to present a counterexample (see [16] for details and proof) which shows that this conjecture is false even when R is regular, and where he defined an R-module M to be *cofinite with respect to* $\mathfrak{a}$ (abbreviated as $\mathfrak{a}$-*cofinite*) if the support of M is contained in $V(\mathfrak{a})$ and $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)$ is finitely generated for all j and asked the following questions:

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(i) *For which rings R and ideals $\mathfrak{a}$ are the modules $H^i_{\mathfrak{a}}(M)$, $\mathfrak{a}$-cofinite for all i and all finitely generated modules M?*
(ii) *Whether the category $\mathscr{C}(R,\mathfrak{a})_{cof}$ of $\mathfrak{a}$-cofinite modules forms an Abelian subcategory of the category of all R-modules?*

With respect to the question (i), Hartshorne in [16] and later Chiriacescu in [9] showed that if R is a complete regular local ring and $\mathfrak{a}$ is a prime ideal such that $\dim R/\mathfrak{a} = 1$, then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for any finitely generated R-module M (see [16, Corollary 7.7]). Also, Delfino and Marley [10, Theorem 1] and Yoshida [25, Theorem 1.1] have eliminated the complete hypothesis entirely. Finally, more recently Bahmanpour and Naghipour removed the local condition on the ring (see [4, Theorem 2.6]).

For a survey of recent developments on finiteness properties of local cohomology modules, see Lyubeznik's interesting paper [17].

In the second section, we establish several characterizations of the $\mathfrak{a}$-cofiniteness of an R-module M. More precisely we prove the following result:

Theorem 1.1. *Let R be a Noetherian ring, M an R-module and $\mathfrak{a}$ a one-dimensional ideal of R such that* $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$. *Then the following conditions are equivalent:*
(i) *M is $\mathfrak{a}$-cofinite.*
(ii) *$H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite, for all i.*
(iii) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is finitely generated, for all $i \leq \mathrm{cd}(\mathfrak{a}, M) + 1$.*
(iv) $\mathrm{Ext}^i_R(N, M)$ *is finitely generated, for all $i \leq \mathrm{cd}(\mathfrak{a}, M) + 1$ and for any finitely generated R-module N with* $\mathrm{Supp}(N) \subseteq V(\mathfrak{a})$.
(v) $\mathrm{Ext}^i_R(N, M)$ *is finitely generated, for all $i \leq \mathrm{cd}(\mathfrak{a}, M) + 1$ and for some finitely generated R-module N with* $\mathrm{Supp}(N) = V(\mathfrak{a})$.

Pursuing this point of view further we derive the following consequence of Theorem 1.1, which is an extension of the main results of Delfino-Marley [10] and Yoshida [25] for an arbitrary Noetherian ring R.

Corollary 1.2. *Let R be a Noetherian ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq$* $\mathrm{Rad}(\mathfrak{a})$. *Let M be a $\mathfrak{b}$-cofinite R-module.*
(i) *If $\dim R/\mathfrak{a} = 1$, then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i.*
(ii) *If $\dim R/\mathfrak{b} = 1$, then $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{a}$-cofinite for all i.*

In [27] H. Zöschinger, introduced the interesting class of minimax modules, and he has in [27, 28] given many equivalent conditions for a module to be minimax. The R-module N is said to be *minimax*, if there is a finitely generated submodule L of N, such that N/L is Artinian. The class of minimax modules thus includes all finitely generated and all Artinian modules. It was shown by T. Zink [26] and by E. Enochs [13] that a module over a complete local ring is minimax if and only if it is Matlis reflexive.

In the second section, we also shall prove the following, which is a generalization of the main result of Brodmann-Lashgari [6].

Theorem 1.3. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module such that for a non-negative integer t, the R-modules $H^i_{\mathfrak{a}}(M)$ are weakly*

Laskerian for all $i \leq t$. Then the R-modules $H^0_{\mathfrak{a}}(M), \ldots, H^t_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite and for any minimax submodule K of $H^{t+1}_{\mathfrak{a}}(M)$ and for any finitely generated R-module L with $\mathrm{Supp}(L) \subseteq V(\mathfrak{a})$, the R-modules $\mathrm{Hom}_R(L, H^{t+1}_{\mathfrak{a}}(M)/K)$ and $\mathrm{Ext}^1_R(L, H^{t+1}_{\mathfrak{a}}(M)/K)$ are finitely generated.

An R-module M is said to be a *weakly Laskerian module*, if the set of associated primes of any quotient module M is finite (see [11] and [23]).

With respect to the question (ii), Hartshorne with an example showed that this not true in general. However, he proved that if $\mathfrak{a}$ is a prime ideal of dimension one in a complete regular local ring R, then the answer to his question is yes. In [10], Delfino and Marley extended this result to arbitrary complete local rings. Recently, Kawasaki [19], by using a spectral sequence argument, generalized the Delfino and Marley's result for an arbitrary ideal $\mathfrak{a}$ of dimension one in a local ring R. Finally, more recently Bahmanpour, Naghipour and Sedghi in [5] removed the local condition on the ring. Namely, therein it is shown that Hartshorne's question is true for $\mathscr{C}^1(R, \mathfrak{a})_{cof}$, the category of all $\mathfrak{a}$-cofinite R-modules M with $\dim \mathrm{Supp}(M) \leq 1$, for all ideals $\mathfrak{a}$ in a Noetherian ring R. The proof of this result is based on [5, Proposition 2.6] which states that in order to deduce the $\mathfrak{a}$-cofiniteness for a module M with $\dim \mathrm{Supp}(M) \leq 1$ and $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$, it suffices that we know that the R-modules $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, M)$ are finitely generated.

The main goal of Section 3 is to establish the analogue of this result to the $\mathfrak{a}$-weakly cofiniteness. Namely, in this section among other things, we show that for the $\mathfrak{a}$-weakly cofiniteness of a module M with $\dim \mathrm{Supp}(M) \leq 1$ and $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$, it suffices that we know that the R-modules $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, M)$ are weakly Laskerian. In particular, when $\mathfrak{a}$ is one-dimensional, in order to deduce the $\mathfrak{a}$-weakly cofiniteness for a module (with support in $V(\mathfrak{a})$), it suffices that we know that the first two Ext-modules in the definition for weakly cofiniteness are weakly Laskerian. More precisely, we shall show that:

Theorem 1.4. *Let $\mathfrak{a}$ denote an ideal of a Noetherian ring R and let M be an R-module such that $\dim \mathrm{Supp}(M) \leq 1$ and $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$. Then M is $\mathfrak{a}$-weakly cofinite if and only if the R-modules $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, M)$ are weakly Laskerian.*

An R-module M is said to be $\mathfrak{a}$-*weakly cofinite* if $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a weakly Laskerian module for all i (see [12]). We denote the category of the $\mathfrak{a}$-weakly cofinite modules by $\mathscr{C}(R, \mathfrak{a})_{wcof}$. As an application of Theorem 1.4 we show that, when $\mathfrak{a}$ is one-dimensional, $\mathscr{C}(R, \mathfrak{a})_{wcof}$ forms an Abelian subcategory of the category of all R-modules (see Corollary 3.6). That is, if $f : M \longrightarrow N$ is an R-homomorphism between $\mathfrak{a}$-weakly cofinite modules, then $\ker f$ and $\mathrm{coker} f$ are $\mathfrak{a}$-weakly cofinite. The proof of this result is based on the following theorem.

Theorem 1.5. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let $\mathscr{C}^1(R, \mathfrak{a})_{wcof}$ denote the category of $\mathfrak{a}$-weakly cofinite R-modules M with $\dim \mathrm{Supp}(M) \leq 1$. Then $\mathscr{C}^1(R, \mathfrak{a})_{wcof}$ is an Abelian category.*

The proof of Theorem 1.5 is given in Theorem 3.5. Finally, we end the paper with a question concerning the Serre subcategory.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ will be an ideal of R. For an R-module M, the i-th local cohomology module of M with support in $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

For facts about the local cohomology modules we refer to the textbook by Brodmann-Sharp [7] or Grothendieck's interesting book [14].

Further, for any ideal $\mathfrak{b}$ of R, we denote the set $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{b}\}$ by $V(\mathfrak{b})$; and the *radical* of $\mathfrak{b}$, denoted by $\operatorname{Rad}(\mathfrak{b})$, we define to be the set $\{x \in R : x^n \in \mathfrak{b} \text{ for some } n \in \mathbb{N}\}$.

For an Artinian R-module A the set of attached prime ideals of A is denoted by $\operatorname{Att}_R A$. Also, for each R-module L, we denote by $\operatorname{Assh}_R L$ the set $\{\mathfrak{p} \in \operatorname{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$. Finally, we shall use $\operatorname{Max}(R)$ to denote the set of all maximal ideals of R. For any unexplained notation and terminology we refer the reader to [8] and [20].

2. Modules cofinite

The main goals of this section are Theorems 2.4 and 2.8. The following lemmas will be needed in the proof of these results. Recall that a class $\mathcal{S}$ of R-modules is a *Serre subcategory* of the category of R-modules, when it is closed under taking submodules, quotients and extensions. It is well known that the subcategories of, finitely generated, minimax, weakly Laskerian, and Matlis reflexive modules are examples of Serre subcategory. Following we let $\mathcal{S}$ denote a Serre subcategory of the category of R-modules.

Lemma 2.1. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let s be a non-negative integer and let M be an R-module such that $\operatorname{Ext}^s_R(R/\mathfrak{a}, M) \in \mathcal{S}$. Suppose that $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $i < s$ and all $j \geq 0$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)) \in \mathcal{S}$.*

Proof. See [1, Theorem 2.2]. □

Lemma 2.2. *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let s be a non-negative integer and let M be an R-module such that $\operatorname{Ext}^{s+1}_R(R/\mathfrak{a}, M) \in \mathcal{S}$. Suppose that $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $i < s$ and all $j \geq 0$. Then $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)) \in \mathcal{S}$.*

Proof. We use induction on s. Let $s = 0$. Then the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow 0, \qquad (\dagger)$$

induces the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, M).$$

As $\operatorname{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, M)$ are in $\mathcal{S}$, it follows that

$$\operatorname{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}.$$

Now, suppose inductively that $s > 0$ and that the assertion holds for $s - 1$. Using the exact sequence $(\dagger)$ we obtain the following exact sequence, $j \geq 0$,

$$\operatorname{Ext}^j_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \longrightarrow \operatorname{Ext}^{j+1}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)).$$

Therefore, since $\operatorname{Ext}^{s+2}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ and $\operatorname{Ext}^{s+1}_R(R/\mathfrak{a}, M)$ are in $\mathcal{S}$, it follows that $\operatorname{Ext}^{s+1}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$. Also, it easily follows from assumption and [7, Corollary 2.1.7] that $\operatorname{Ext}^{j}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))) \in \mathcal{S}$ for all $i < s$ and all $j \geq 0$. Therefore we may assume that $\Gamma_{\mathfrak{a}}(M) = 0$.

Next, let $E_R(M)$ denote the injective hull of M. Then $\Gamma_{\mathfrak{a}}(E_R(M)) = 0$, and so it follows from the exact sequence

$$0 \longrightarrow M \longrightarrow E_R(M) \longrightarrow E_R(M)/M \longrightarrow 0,$$

that $H^{i+1}_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(E_R(M)/M)$ for all $i \geq 0$. Also, as $\operatorname{Hom}_R(R/\mathfrak{a}, E_R(M)) = 0$, it yields that

$$\operatorname{Ext}^{j}_R(R/\mathfrak{a}, M) \cong \operatorname{Ext}^{j+1}_R(R/\mathfrak{a}, M),$$

for all $j \geq 0$. Consequently the R-module $E_R(M)/M$ satisfies our condition hypothesis. Thus $\operatorname{Ext}^1_R(R/\mathfrak{a}, H^{s-1}_{\mathfrak{a}}(E_R(M)/M)) \in \mathcal{S}$. Now the assertion follows from the isomorphism

$$H^s_{\mathfrak{a}}(M) \cong H^{s-1}_{\mathfrak{a}}(E_R(M)/M).$$

□

Lemma 2.3. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M a non-zero R-module, such that* $\dim \operatorname{Supp}(M) \leq 1$ *and* $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$. *Then the following statements are equivalent:*

(i) *M is $\mathfrak{a}$-cofinite.*

(ii) *The R-modules* $\operatorname{Hom}_R(R/\mathfrak{a}, M)$ *and* $\operatorname{Ext}^1_R(R/\mathfrak{a}, M)$ *are finitely generated.*

Proof. See [5, Proposition 2.6]. □

Now we are prepared to state and prove the first main theorem of this section. Recall that for an R-module N, the *cohomological dimension of N with respect to* an ideal $\mathfrak{a}$ of R, denoted by $\operatorname{cd}(\mathfrak{a}, N)$, is defined as

$$\operatorname{cd}(\mathfrak{a}, N) = \sup\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(N) \neq 0\}.$$

Theorem 2.4. *Let R be a Noetherian ring, M an R-module and $\mathfrak{a}$ a one-dimensional ideal of R. Then the following conditions are equivalent:*

(i) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is finitely generated, for all* $i \leq \operatorname{cd}(\mathfrak{a}, M) + 1$.

(ii) $H^i_{\mathfrak{a}}(M)$ *is $\mathfrak{a}$-cofinite, for all i.*

(iii) $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ *is finitely generated, for all i.*

(iv) $\operatorname{Ext}^i_R(N, M)$ *is finitely generated, for all* $i \leq \operatorname{cd}(\mathfrak{a}, M) + 1$ *and for any finitely generated R-module N with* $\operatorname{Supp}(N) \subseteq V(\mathfrak{a})$.

(v) $\operatorname{Ext}^i_R(N, M)$ *is finitely generated, for all* $i \leq \operatorname{cd}(\mathfrak{a}, M) + 1$ *and for some finitely generated R-module N with* $\operatorname{Supp}(N) = V(\mathfrak{a})$.

(vi) $\operatorname{Ext}^i_R(N, M)$ *is finitely generated, for all i and for any finitely generated R-module N with* $\operatorname{Supp}(N) \subseteq V(\mathfrak{a})$.

(vii) $\operatorname{Ext}^i_R(N, M)$ *is finitely generated, for all i and for some finitely generated R-module N with* $\operatorname{Supp}(N) = V(\mathfrak{a})$.

Proof. In order to prove (i) $\Longrightarrow$ (ii) we may assume that $i \leq \mathrm{cd}(\mathfrak{a}, M)$. Now, we use induction on i. When $i = 0$, then the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow M/\Gamma_{\mathfrak{a}}(M) \longrightarrow 0,$$

induces the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, M) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))$$
$$\longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, M).$$

As $\mathrm{Hom}_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)) = 0$ and $\mathrm{Ext}^j_R(R/\mathfrak{a}, M)$, for $j = 0, 1$, is finitely generated, it follows that $\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$ are finitely generated. It now follows from Lemma 2.3 that $\Gamma_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite.

Assume, inductively, that $i > 0$ and that the result has been proved for $i - 1$. Then the R-modules

$$H^0_{\mathfrak{a}}(M), H^1_{\mathfrak{a}}(M), \ldots, H^{i-1}_{\mathfrak{a}}(M),$$

are $\mathfrak{a}$-cofinite, and so it follows from Lemmas 2.1 and 2.2 that $\mathrm{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ are finitely generated. Now, it yields from Lemma 2.3 that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite.

The implication (ii) $\Longrightarrow$ (iii) follows from [22, Proposition 3.9], and for prove (iii) $\Longrightarrow$ (vi) see [18, Lemma 1]. Finally, in order to complete the proof, it is enough for us to show that (v) $\Longrightarrow$ (iv). To this end, let L be a finitely generated R-module with $\mathrm{Supp}(L) \subseteq V(\mathfrak{a})$ and N a finitely generated R-module such that $\mathrm{Supp}(N) = V(\mathfrak{a})$. Then $\mathrm{Supp}(L) \subseteq \mathrm{Supp}(N)$, and so according to Gruson's Theorem [24, Theorem 4.1], there exists a chain

$$0 = L_0 \subset L_1 \subset \cdots \subset L_k = L,$$

such that the factors L_j/L_{j-1} are homomorphic images of a direct sum of finitely many copies of N. Now consider the exact sequences

$$0 \longrightarrow K \longrightarrow N^n \longrightarrow L_1 \longrightarrow 0$$

$$0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow L_2/L_1 \longrightarrow 0$$
$$\vdots$$
$$0 \longrightarrow L_{k-1} \longrightarrow L_k \longrightarrow L_k/L_{k-1} \longrightarrow 0,$$

for some positive integer n. Now, from the long exact sequence

$$\cdots \to \mathrm{Ext}^{i-1}_R(L_{j-1}, N) \to \mathrm{Ext}^i_R(L_j/L_{j-1}, N) \to \mathrm{Ext}^i_R(L_j, N) \to \mathrm{Ext}^i_R(L_{j-1}, N) \to \cdots,$$

and an easy induction on k, it suffices to prove the case when $k = 1$.

Thus there is an exact sequence

$$0 \longrightarrow K \longrightarrow N^n \longrightarrow L \longrightarrow 0 \qquad (\dagger)$$

for some $n \in \mathbb{N}$ and some finitely generated R-module K.

Now, we use induction on i. First, $\mathrm{Hom}_R(L, M)$ is a submodule of $\mathrm{Hom}_R(N^n, M)$; hence, in view of assumption, $\mathrm{Ext}^0_R(L, M)$ is finitely generated. So assume that $i > 0$

and that $\operatorname{Ext}^j_R(L', M)$ is finitely generated for every finitely generated R-module L' with $\operatorname{Supp}(L') \subseteq \operatorname{Supp}(N)$ and for all $j \leq i-1$. Now, the exact sequence (†) induces the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}_R(K, M) \longrightarrow \operatorname{Ext}^i_R(L, M) \longrightarrow \operatorname{Ext}^i_R(N^n, M) \longrightarrow \cdots,$$

so that, by the inductive hypothesis, $\operatorname{Ext}^{i-1}_R(K, M)$ is finitely generated. On the other hand $\operatorname{Ext}^i_R(N^n, M) \cong \overset{n}{\oplus} \operatorname{Ext}^i_R(L, M)$ is finitely generated, and so $\operatorname{Ext}^i_R(L, M)$ is finitely generated, the inductive step is complete. ☐

As a consequence of Theorem 2.4, we derive the following result which is an extension of the main results of Delfino-Marley [10] and Yoshida [25] for arbitrary Noetherian rings.

Corollary 2.5. *Let R be a Noetherian ring and $\mathfrak{a}, \mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq \operatorname{Rad}(\mathfrak{a})$. Let M be a $\mathfrak{b}$-cofinite R-module.*

(i) *If $\dim R/\mathfrak{a} = 1$, then the R-module $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i.*

(ii) *If $\dim R/\mathfrak{b} = 1$, then the R-module $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{a}$-cofinite for all i.*

Proof. In order to show (i), since $\mathfrak{b} \subseteq \operatorname{Rad}(\mathfrak{a})$, it follows that $\operatorname{Supp}(R/\mathfrak{a}) \subseteq \operatorname{Supp}(R/\mathfrak{b})$. On the other hand, since M is $\mathfrak{b}$-cofinite it follows from [18, Lemma 1] that M is also $\mathfrak{a}$-cofinite. Now as $\dim R/\mathfrak{a} = 1$, it follows from Theorem 2.4 that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i.

To prove (ii), since $\dim R/\mathfrak{b} = 1$ and M is $\mathfrak{b}$-cofinite it follows from Theorem 2.4 that $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite for all i. Now, because of $\operatorname{Supp}(R/\mathfrak{a}) \subseteq \operatorname{Supp}(R/\mathfrak{b})$ it follows from [18, Lemma 1] that $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{a}$-cofinite, for all i. ☐

Before proving the next main theorem, we need the following lemma and proposition, which will be used in Theorem 2.8.

Lemma 2.6. *Let R be a Noetherian ring and M an R-module. Then M is weakly Laskerian if and only if there exists a finitely generated submodule N of M such that $\operatorname{Supp}(M)/N$ is finite.*

Proof. See [2, Theorem 3.3]. ☐

Proposition 2.7. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module such that $H^i_{\mathfrak{a}}(M)$ is weakly Laskerian for all $i \leq t$. Then the R-modules*

$$H^0_{\mathfrak{a}}(M), \ldots, H^t_{\mathfrak{a}}(M)$$

are $\mathfrak{a}$-cofinite. In addition the R-modules

$$\operatorname{Hom}_R(R/\mathfrak{a}, H^{t+1}_{\mathfrak{a}}(M)) \text{ and } \operatorname{Ext}^1_R(R/\mathfrak{a}, H^{t+1}_{\mathfrak{a}}(M))$$

are finitely generated. In particular, the set $\operatorname{Ass}_R H^{t+1}_{\mathfrak{a}}(M)$ is finite.

Proof. We use induction on t. The case $t = 0$ follows from Lemmas 2.1 and 2.2. So, let $t \geq 1$ and the case $t-1$ is settled. Then by inductive hypothesis the R-modules $H^0_{\mathfrak{a}}(M), \ldots, H^{t-1}_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite and the R-modules

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^t_\mathfrak{a}(M)) \text{ and } \mathrm{Ext}^1_R(R/\mathfrak{a}, H^t_\mathfrak{a}(M))$$

are finitely generated. Now since by assumption the R-module $H^t_\mathfrak{a}(M)$ is weakly Laskerian, it follows from Lemma 2.6 that there is a finitely generated submodule N of $H^t_\mathfrak{a}(M)$ such that $\mathrm{Supp}(H^t_\mathfrak{a}(M)/N)$ is finite set, and so $\dim \mathrm{Supp}(H^t_\mathfrak{a}(M)/N) \leq 1$. Now it follows from the exact sequence

$$0 \longrightarrow N \longrightarrow H^t_\mathfrak{a}(M) \longrightarrow H^t_\mathfrak{a}(M)/N \longrightarrow 0,$$

that the R-modules

$$\mathrm{Hom}_R(R/\mathfrak{a}, H^t_\mathfrak{a}(M)/N) \quad \text{and} \quad \mathrm{Ext}^1_R(R/\mathfrak{a}, H^t_\mathfrak{a}(M)/N),$$

are finitely generated. Therefore it follows from Lemma 2.3 that the R-module $H^t_\mathfrak{a}(M)/N$ is $\mathfrak{a}$-cofinite, and so the R-module $H^t_\mathfrak{a}(M)$ is $\mathfrak{a}$-cofinite. Hence, it follows from Lemmas 2.1 and 2.2 that the R-modules $\mathrm{Hom}_R(R/\mathfrak{a}, H^{t+1}_\mathfrak{a}(M))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, H^{t+1}_\mathfrak{a}(M))$ are finitely generated. This completes the induction step. □

Now, we are ready to state and prove the second main result of this section, which is a generalization the main results of Bahmanpour-Naghipour [3, Theorem 2.6] and Brodmann-Lashgari [6, Theorem 2.2].

Theorem 2.8. *Let R be a Noetherian ring, $\mathfrak{a}$ an ideal of R and M a finitely generated R-module such that for a non-negative integer t, the R-modules $H^i_I(M)$ are weakly Laskerian for all $i \leq t$. Then the R-modules*

$$H^0_\mathfrak{a}(M), \dots, H^t_\mathfrak{a}(M)$$

are $\mathfrak{a}$-cofinite and for any minimax submodule K of $H^{t+1}_\mathfrak{a}(M)$ and for any finitely generated R-module L with $\mathrm{Supp}(L) \subseteq V(\mathfrak{a})$, *the R-modules*

$$\mathrm{Hom}_R(L, H^{t+1}_\mathfrak{a}(M)/K) \text{ and } \mathrm{Ext}^1_R(L, H^{t+1}_\mathfrak{a}(M)/K)$$

are finitely generated.

Proof. By virtue of Proposition 2.7 the R-module $H^i_\mathfrak{a}(M)$ is $\mathfrak{a}$-cofinite for all $i \leq t$ and $\mathrm{Hom}_R(R/\mathfrak{a}, H^{t+1}_\mathfrak{a}(M))$ is finitely generated. Hence the R-module $\mathrm{Hom}_R(R/\mathfrak{a}, K)$ is finitely generated, and so in view of [22, Proposition 4.3], K is $\mathfrak{a}$-cofinite. Thus, [18, Lemma 1] implies that $\mathrm{Ext}^i_R(L, K)$ is finitely generated for all i.

Next, the exact sequence

$$0 \longrightarrow K \longrightarrow H^{t+1}_\mathfrak{a}(M) \longrightarrow H^{t+1}_\mathfrak{a}(M)/K \longrightarrow 0$$

provides the following exact sequence,

$$\begin{aligned} &\mathrm{Hom}_R(L, \mathrm{H}^{t+1}_\mathfrak{a}(M)) \longrightarrow \mathrm{Hom}_R(L, H^{t+1}_\mathfrak{a}(M)/K) \longrightarrow \mathrm{Ext}^1_R(L, K) \\ &\longrightarrow \mathrm{Ext}^1_R(L, H^{t+1}_\mathfrak{a}(M)) \longrightarrow \mathrm{Ext}^1_R(L, H^{t+1}_\mathfrak{a}(M)/K) \longrightarrow \mathrm{Ext}^2_R(L, K). \end{aligned}$$

Now, since $\mathrm{Ext}^i_R(L, K)$ is finitely generated, the assertion follows from Proposition 2.7 and [18, Lemma 1], because the R-modules

$$\mathrm{Hom}_R(L, H^{t+1}_\mathfrak{a}(M)) \text{ and } \mathrm{Ext}^1_R(L, H^{t+1}_\mathfrak{a}(M))$$

are finitely generated. □

3. Modules weakly cofinite

The purpose of this section is to establish that the category of modules weakly cofinite with respect to an ideal of dimension one in a Noetherian ring is a full Abelian subcategory of the category of modules. The main goal of this section is Theorem 3.5. The proof of this theorem is based on the Proposition 3.2, which plays a key role in this section, says that (when $\mathfrak{a}$ is one-dimensional), in order to deduce the $\mathfrak{a}$-weakly cofiniteness for a module (with support in $V(\mathfrak{a})$), it suffices that we know that the first two Ext-modules in the definition for weakly cofiniteness are weakly Laskerian. Before stating it, we record a lemma that will be needed in the proof of this proposition.

Lemma 3.1. *Let $(R, \mathfrak{m})$ be a local (Noetherian) ring and let A be an Artinian R-module.*
(i) *If $\mathfrak{a}$ is an ideal of R such that $\operatorname{Hom}_R(R/\mathfrak{a}, A)$ is a finitely generated R-module, then*

$$V(\mathfrak{a}) \cap \operatorname{Att}_R A \subseteq V(\mathfrak{m}).$$

(ii) *If x is an element of R such that $V(Rx) \cap \operatorname{Att}_R A \subseteq \{\mathfrak{m}\}$, then the R-module A/xA has finite length.*

Proof. See [4, Lemmas 2.4 and 2.5]. □

The following proposition will be one our main tools in this section. It's proof is based on the important notion of the arithmetic rank of an ideal. The *arithmetic rank* of an ideal $\mathfrak{b}$ in a Noetherian ring R, denoted by $\operatorname{ara}(\mathfrak{b})$, is the least number of elements of R required to generate an ideal which has the same radical as $\mathfrak{b}$, i.e.,

$$\operatorname{ara}(\mathfrak{b}) = \min\{n \in \mathbb{N}_0 : \exists b_1, \ldots, b_n \in R \text{ with } \operatorname{Rad}(b_1, \ldots, b_n) = \operatorname{Rad}(\mathfrak{b})\}.$$

Let M be an R-module. The arithmetic rank of an ideal $\mathfrak{b}$ of R with respect to M, denoted by $\operatorname{ara}_M(\mathfrak{b})$, is defined the arithmetic rank of the ideal $\mathfrak{b} + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)$ in the ring $R/\operatorname{Ann}_R(M)$.

Proposition 3.2. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M an R-module such that $\dim \operatorname{Supp}(M) \leq 1$ and $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$. Then the following statements are equivalent:*
(i) *M is $\mathfrak{a}$-weakly cofinite.*
(ii) *The R-modules $\operatorname{Hom}_R(R/\mathfrak{a}, M)$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, M)$ are weakly Laskerian.*

Proof. The conclusion (i) $\Longrightarrow$ (ii) is obviously true. In order to prove that (ii) $\Longrightarrow$ (i), as

$$\operatorname{Ass}_R \operatorname{Hom}_R(R/\mathfrak{a}, M) = \operatorname{Ass}_R M$$

and $\operatorname{Hom}_R(R/\mathfrak{a}, M)$ is weakly Laskerian, it follows that $\operatorname{Ass}_R M$ is finite. Now, if $\dim \operatorname{Supp}(M) = 0$, then $\operatorname{Ass}_R M = \operatorname{Supp}(M)$, and so $\operatorname{Supp}(M)$ is also finite. Therefore, in view of definition, M is weakly Laskerian, and so by [12, Lemma 2.2], M is $\mathfrak{a}$-weakly cofinite. Consequently, we may assume $\dim \operatorname{Supp}(M) = 1$; and we use induction on

$$t := \operatorname{ara}_M(\mathfrak{a}) = \operatorname{ara}(\mathfrak{a} + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M))$$

that M is $\mathfrak{a}$-weakly cofinite. If $t=0$, then it follows from definition that $\mathfrak{a}^n \subseteq \operatorname{Ann}_R(M)$ for some positive integer n, and so $M=(0:_M \mathfrak{a}^n)$. Therefore the assertion follows from [12, Lemma 2.8]. So assume that $t>0$ and the result has been proved for all $i \leq t-1$. In view of Lemma 2.6 there exist finitely generated submodules A of $\operatorname{Hom}_R(R/\mathfrak{a}, M)$ and B of $\operatorname{Ext}^1_R(R/\mathfrak{a}, M)$ such that the set

$$\Omega := \operatorname{Supp}(\operatorname{Hom}_R(R/\mathfrak{a}, M)/A) \bigcup \operatorname{Supp}(\operatorname{Ext}^1_R(R/\mathfrak{a}, M)/B).$$

is finite. Now, let

$$\mathscr{T} = \{\mathfrak{p} \in \operatorname{Supp}(M) \mid \dim R/\mathfrak{p} = 1\} \setminus \Omega.$$

It is easy to see that $\mathscr{T} \subseteq \operatorname{Assh}_R M$, and so $\mathscr{T}$ is finite. (Note that $\operatorname{Ass}_R M$ is finite.)

In addition, as $\Omega \subseteq \operatorname{Supp}(M)$, it follows that

$$\max\{\dim \operatorname{Supp}(\operatorname{Hom}_R(R/\mathfrak{a}, M)/A), \dim \operatorname{Supp}(\operatorname{Ext}^1_R(R/\mathfrak{a}, M)/B)\} \leq 1.$$

Therefore, in view of the prime avoidance theorem it is easy to see that, for each $\mathfrak{p} \in \mathscr{T}$ we have $\mathfrak{p} \not\subseteq \bigcup_{\mathfrak{q} \in \Omega} \mathfrak{q}$. Consequently, it is easily yields that

$$(\operatorname{Hom}_R(R/\mathfrak{a}, M)/A)_{\mathfrak{p}} = 0 = (\operatorname{Ext}^1_R(R/\mathfrak{a}, M)/B)_{\mathfrak{p}}.$$

Whence for each $\mathfrak{p} \in \mathscr{T}$ the $R_{\mathfrak{p}}$-module $\operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}})$ is finitely generated, by [20, Ex. 7.7], and $M_{\mathfrak{p}}$ is an $\mathfrak{a}R_{\mathfrak{p}}$-torsion $R_{\mathfrak{p}}$-module, with $\operatorname{Supp}(M)_{\mathfrak{p}} \subseteq V(\mathfrak{p}R_{\mathfrak{p}})$, and so it follows that the $R_{\mathfrak{p}}$-module $\operatorname{Hom}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}})$ is Artinian. Consequently, according to Melkersson's results [21, Theorem 1.3] and [22, Proposition 4.3], $M_{\mathfrak{p}}$ is an Artinian and $\mathfrak{a}R_{\mathfrak{p}}$-cofinite $R_{\mathfrak{p}}$-module. Next, let $\mathscr{T} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Then by Lemma 3.1(i), we have

$$V(\mathfrak{a}R_{\mathfrak{p}_j}) \cap \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) \subseteq V(\mathfrak{p}_j R_{\mathfrak{p}_j}),$$

for all $j = 1, 2, \dots, n$. Next, set

$$\mathscr{U} := \bigcup_{j=1}^n \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})\}.$$

It is easy to check that $\mathscr{U} \cap V(\mathfrak{a}) \subseteq \mathscr{T}$.

On the other hand, since $t = \operatorname{ara}_M(\mathfrak{a}) \geq 1$, there exist elements $y_1, \dots, y_t \in \mathfrak{a}$ such that

$$\operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) = \operatorname{Rad}((y_1, \dots, y_t) + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)).$$

Now, as $\mathfrak{a} \not\subseteq \bigcup_{\mathfrak{q} \in \mathscr{U} \setminus V(\mathfrak{a})} \mathfrak{q}$, it follows that $(y_1, \dots, y_t) + \operatorname{Ann}_R(M) \not\subseteq \bigcup_{\mathfrak{q} \in \mathscr{U} \setminus V(\mathfrak{a})} \mathfrak{q}$.

Furthermore, for each $\mathfrak{q} \in \mathscr{U}$ we have $\mathfrak{q}R_{\mathfrak{p}_j} \in \operatorname{Att}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j})$, for some integer $1 \leq j \leq n$. Whence

$$\operatorname{Ann}_R(M)R_{\mathfrak{p}_j} \subseteq \operatorname{Ann}_{R_{\mathfrak{p}_j}}(M_{\mathfrak{p}_j}) \subseteq \mathfrak{q}R_{\mathfrak{p}_j}.$$

Since $\mathfrak{q}$ is prime we get that $\operatorname{Ann}_R(M) \subseteq \mathfrak{q}$. Consequently, it follows from

$$\operatorname{Ann}_R(M) \subseteq \bigcap_{\mathfrak{q} \in \mathscr{U} \setminus V(\mathfrak{a})} \mathfrak{q},$$

that $(y_1, \dots, y_t) \not\subseteq \bigcup_{\mathfrak{q} \in \mathscr{U} \setminus V(\mathfrak{a})} \mathfrak{q}$. Therefore, by [20, Ex. 16.8] there is $a \in (y_2, \dots, y_t)$ such that $y_1 + a \notin \bigcup_{\mathfrak{q} \in \mathscr{U} \setminus V(\mathfrak{a})} \mathfrak{q}$. Let $x := y_1 + a$. Then $x \in \mathfrak{a}$ and

$$\operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)) = \operatorname{Rad}((x, y_2, ..., y_t) + \operatorname{Ann}_R(M)/\operatorname{Ann}_R(M)).$$

Next, let $N := (0 :_M x)$. Then, it is easy to see that

$$\operatorname{ara}_N(\mathfrak{a}) = \operatorname{ara}(\mathfrak{a} + \operatorname{Ann}_R(N)/\operatorname{Ann}_R(N)) \leq t-1.$$

(note that $x \in \operatorname{Ann}_R N$), and hence

$$\operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R(N)/\operatorname{Ann}_R(N)) = \operatorname{Rad}((y_2, \ldots, y_t) + \operatorname{Ann}_R(N)/\operatorname{Ann}_R(N))).$$

Now, the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow xM \longrightarrow 0, \qquad (\dagger)$$

induces an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, N) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, xM)$$
$$\longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, N) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, M),$$

which implies that the R-modules $\operatorname{Hom}_R(R/\mathfrak{a}, N)$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, N)$ are weakly Laskerian. Consequently, by the inductive hypothesis, the R-module N is $\mathfrak{a}$-weakly cofinite.

Moreover, the exact sequence ($\dagger$) induces the exact sequence

$$\operatorname{Ext}^1_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, xM) \longrightarrow \operatorname{Ext}^2_R(R/\mathfrak{a}, N),$$

which implies that the R-module $\operatorname{Ext}^1_R(R/\mathfrak{a}, xM)$ is weakly Laskerian.

Also, from the exact sequence

$$0 \longrightarrow xM \longrightarrow M \longrightarrow M/xM \longrightarrow 0$$

we get the exact sequence

$$\operatorname{Hom}_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, M/xM) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, xM)$$

which implies that the R-module $\operatorname{Hom}_R(R/\mathfrak{a}, M/xM)$ is weakly Laskerian.
Now, from Lemma 3.1(ii), it is easy to see that the $R_{\mathfrak{p}_j}$-module $(M/xM)_{\mathfrak{p}_j}$ has finite length for all $j = 1, \ldots, n$. Therefore there exists a finitely generated submodule L_j of M/xM such that

$$(M/xM)_{\mathfrak{p}_j} = (L_j)_{\mathfrak{p}_j}.$$

Let $L := L_1 + \cdots + L_n$. Then L is a finitely generated submodule of M/xM such that

$$\operatorname{Supp}_R(M/xM)/L \subseteq \operatorname{Supp}(M) \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subseteq (\operatorname{Ass}_R M \bigcap \operatorname{Max}(R)) \bigcup \Omega.$$

The exact sequence

$$0 \longrightarrow L \longrightarrow M/xM \longrightarrow (M/xM)/L \longrightarrow 0,$$

provides the following exact sequence,

$$\operatorname{Hom}_R(R/\mathfrak{a}, M/xM) \longrightarrow \operatorname{Hom}_R(R/\mathfrak{a}, (M/xM)/L) \longrightarrow \operatorname{Ext}^1_R(R/\mathfrak{a}, L);$$

which implies that $\operatorname{Hom}_R(R/\mathfrak{a}, (M/xM)/L)$ is weakly Laskerian.

We now show that M/xM is a weakly Laskerian R-module. To do this, since the sets $\operatorname{Ass}_R M \cap \operatorname{Max}(R)$ and Ω are finite, it follows that the set $\operatorname{Supp}(M/xM)/L$ is finite too. Thus, as L is finitely generated, it follows from Lemma 2.6 that M/xM is a weakly

Laskerian R-module. Thus in view of [12, Lemma 2.6] the R-module M/xM is a $\mathfrak{a}$-weakly cofinite. Now, since the R-modules $N = (0 :_M x)$ and M/xM are $\mathfrak{a}$-weakly cofinite, it follows from [22, Lemma 3.1] and [12, Lemma 2.2] that M is $\mathfrak{a}$-weakly cofinite module. This completes the inductive step. □

The first application of Proposition 3.2 gives us a characterization of the $\mathfrak{a}$-weakly cofiniteness of an R-module M in terms of the $\mathfrak{a}$-weakly cofiniteness of the local cohomology modules $H^i_{\mathfrak{a}}(M)$.

Corollary 3.3. *Let R be a Noetherian ring, M an R-module and $\mathfrak{a}$ a one-dimensional ideal of R. Then the following conditions are equivalent:*

(i) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for all* $i \leq \mathrm{cd}(\mathfrak{a}, M) + 1$.

(ii) $H^i_{\mathfrak{a}}(M)$ *is $\mathfrak{a}$-weakly cofinite for all i.*

(iii) $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ *is weakly Laskerian for all i.*

(iv) $\mathrm{Ext}^i_R(N, M)$ *is weakly Laskerian for all* $i \leq \mathrm{cd}(\mathfrak{a}, M) + 1$ *and for any finitely generated R-module N with* $\mathrm{Supp}(N) \subseteq V(\mathfrak{a})$.

(v) $\mathrm{Ext}^i_R(N, M)$ *is weakly Laskerian for all* $i \leq \mathrm{cd}(\mathfrak{a}, M) + 1$ *and for some finitely generated R-module N with* $\mathrm{Supp}(N) = V(\mathfrak{a})$.

(vi) $\mathrm{Ext}^i_R(N, M)$ *is weakly Laskerian for all i and for any finitely generated R-module N with* $\mathrm{Supp}(N) \subseteq V(\mathfrak{a})$.

(vii) $\mathrm{Ext}^i_R(N, M)$ *is weakly Laskerian for all i and for some finitely generated R-module N with* $\mathrm{Supp}(N) = V(\mathfrak{a})$.

Proof. By a slight modification of the proof of Theroem 2.4, the result follows easily from Proposition 3.2 and Lemmas 2.1, 2.2, by applying [12, Lemmas 2.2 and 2.8]. □

Corollary 3.4. *Let R be a Noetherian ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals of R such that $\mathfrak{b} \subseteq$ $\mathrm{Rad}(\mathfrak{a})$. Let M be a $\mathfrak{b}$-weakly cofinite R-module.*

(i) *If* $\dim R/\mathfrak{a} = 1$, *then the R-module $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i.*

(ii) *If* $\dim R/\mathfrak{b} = 1$, *then the R-module $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i.*

Proof. In order to show that (i), since $\mathfrak{b} \subseteq \mathrm{Rad}(\mathfrak{a})$, it follows that $\mathrm{Supp}(R/\mathfrak{a}) \subseteq \mathrm{Supp}(R/\mathfrak{b})$. On the other hand, since M is $\mathfrak{b}$-weakly cofinite it follows from [12, Lemma 2.8] that M is also $\mathfrak{a}$-weakly cofinite. Now since $\dim R/\mathfrak{a} = 1$, the result follows from Corollary 3.3.

To prove (ii), since $\dim R/\mathfrak{b} = 1$ and M is $\mathfrak{b}$-weakly cofinite it follows from Corollary 3.3 that $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-weakly cofinite for all i. Now as $\mathrm{Supp}(R/\mathfrak{a}) \subseteq \mathrm{Supp}(R/\mathfrak{b})$ it follows from [12, Lemma 2.8] that $H^i_{\mathfrak{b}}(M)$ is $\mathfrak{a}$-weakly cofinite for all i. □

We are now in a position to use Proposition 3.2 to produce a proof of the main theorem of this section, which shows that $\mathscr{C}^1(R, \mathfrak{a})_{wcof}$, the category of $\mathfrak{a}$-weakly cofinite R-modules M with $\dim \mathrm{Supp}(M) \leq 1$, is a full Abelian subcategory of the category of modules.

Theorem 3.5. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R. Let $\mathscr{C}^1(R, \mathfrak{a})_{wcof}$ denote the category of $\mathfrak{a}$-weakly cofinite R-modules M with* $\dim \mathrm{Supp}(M) \leq 1$. *Then $\mathscr{C}^1(R, \mathfrak{a})_{wcof}$ is an Abelian category.*

Proof. Let $M, N \in \mathscr{C}^1(R,\mathfrak{a})_{wcof}$ and let $f : M \longrightarrow N$ be an R-homomorphism. We show that the R-modules $\ker f$ and $\mathrm{coker} f$ are $\mathfrak{a}$-weakly cofinite. To this end, the exact sequence

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \mathrm{im} f \longrightarrow 0,$$

induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \ker f) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, M) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{im} f)$$
$$\longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, \ker f) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, M),$$

that implies the R-modules

$$\mathrm{Hom}_R(R/\mathfrak{a}, \ker f) \text{ and } \mathrm{Ext}^1_R(R/\mathfrak{a}, \ker f),$$

are weakly cofinite. Therefore it follows from Proposition 3.2 that $\ker f$ is $\mathfrak{a}$-weakly cofinite. Now, by using the exact sequences

$$0 \longrightarrow \ker f \longrightarrow M \longrightarrow \mathrm{im} f \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{im} f \longrightarrow N \longrightarrow \mathrm{coker} f \longrightarrow 0,$$

we see that $\mathrm{coker} f$ is also $\mathfrak{a}$-weakly cofinite, as required. □

As an immediate consequence of Theorem 3.5, we derive the weakly cofiniteness version of Delfino-Marley's result in [10] and Kawasaki's result in [19], which shows that the category of modules weakly cofinite, with respect to an ideal of dimension one in a Noetherian ring, is a full Abelian subcategory of the category of modules. Following, we let $\mathscr{C}(R,\mathfrak{a})_{wcof}$ denote the category of modules weakly cofinite with respect to $\mathfrak{a}$.

Corollary 3.6. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R of dimension one. Then $\mathscr{C}(R,\mathfrak{a})_{wcof}$ forms an Abelian subcategory of the category of all R-modules.*

Proof. As $\mathrm{Supp}(M) \subseteq \mathrm{Supp}(R/\mathfrak{a})$ for all $M \in \mathscr{C}(R,\mathfrak{a})_{wcof}$, and $\dim R/\mathfrak{a} = 1$, it follows that $\dim \mathrm{Supp}(M) \leq 1$. Now the assertion follows from Theorem 3.5. □

Corollary 3.7. *Let $\mathfrak{a}$ be an ideal of a Noetherian ring R of dimension one. Let*

$$X^\bullet : \cdots \longrightarrow X^i \xrightarrow{f^i} X^{i+1} \xrightarrow{f^{i+1}} X^{i+2} \longrightarrow \cdots,$$

be a complex such that $X^i \in \mathscr{C}(R,\mathfrak{a})_{wcof}$ for all $i \in \mathbb{Z}$. Then the i-th homology module $H^i(X^\bullet)$ is in $\mathscr{C}(R,\mathfrak{a})_{wcof}$.

Proof. The assertion follows from Corollary 3.6. □

Corollary 3.8. *Let $\mathfrak{a} = (x_1, \ldots, x_n)$ be an ideal of a Noetherian ring R. Let M and N be two R-modules such that N is finitely generated and M is $\mathfrak{a}$-weakly cofinite with* $\dim \mathrm{Supp}(M) \leq 1$. *Then the R-modules $\mathrm{Ext}^i_R(N, M)$, $\mathrm{Tor}^R_i(N, M)$ and; the Koszul homology module $H_i(x_1, \ldots, x_n; M)$ are $\mathfrak{a}$-weakly cofinite for all i.*

Proof. By considering a finite free resolution $\mathcal{F}_\bullet \longrightarrow N$ of N, and applying Theorem 3.5 to the complexes

$$\mathrm{Hom}(\mathcal{F}_\bullet, M), \quad \mathcal{F}_\bullet \otimes_R M, \quad \mathbf{K}_\bullet(x_1, \ldots, x_n; M),$$

the assertion follows. □

We end the paper with the following question:

Question. Let $\mathfrak{a}$ be an ideal of a Noetherian ring R and M an R-module such that $\dim \mathrm{Supp}(M) \leq 1$ and $\mathrm{Supp}(M) \subseteq V(\mathfrak{a})$. Let $\mathcal{S}$ be a Serre subcategory of the category of R-modules. Is the following statements are equivalent ?
(i) The R-modules $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ are in $\mathcal{S}$, for all $i \geq 0$.
(ii) The R-modules $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, M)$ are in $\mathcal{S}$.

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